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Sphere Emitting Charged Null Fluid in Einstein's Universe

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Solutions of the Einstein–Maxwell equations with the addition of terms representing charged null fluid emitted from a spherically symmetric body and perfect fluid are obtained. The solutions of Tupper and Patel and Akabari are derived as particular cases.

1. INTRODUCTION

Patel and Akabari (1979) have transformed the metric of Einstein's universe

$$(ds)^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{(x\,dx + y\,dy + z\,dz)^2}{b^2 - (x^2 + y^2 + z^2)} \tag{1}$$

into the form

$$(ds)^2 = 2du\,dr + du^2 - b^2\sin^2(r/b)(d\alpha^2 + \sin^2\alpha\,d\beta^2) \tag{2}$$

where b is a constant.

Tupper (1974) has obtained solutions of the Einstein–Maxwell equations with the addition of terms representing charged null fluid emitted from a spherically symmetric body. The geometry of these solutions is described by the metric

$$(ds)^2 = 2du\,dr + B\,du^2 - r^2(d\alpha^2 + \sin^2\alpha\,d\beta^2) \tag{3}$$

where B is of the form

$$B = 1 - \frac{2m(u)}{r} + \frac{h(u)}{r^2} \tag{4}$$

m and h being arbitrary functions of u.

In the absence of the source (i.e., when $m = h = 0$), the metric (3) becomes flat. Thus the metric (3) is described under the flat background. It would be interesting to obtain the metric (3) in the cosmological background of Einstein's universe rather than the standard Minkowskian background. The object of the present paper is to do just that.

For this purpose we consider the line element

$$(ds)^2 = 2du\,dr + 2L(du)^2 - b^2 \sin^2(r/b)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \tag{5}$$

where L is a function of r and u.

In this paper we find all the solutions of the form (5) for the field equations used by Patel and Akabari, which are

$$R_i^k - (1/2)\delta_i^k R = -8\pi\left[E_i^k + \sigma v_i v^k + (p+\rho)V_i V^k - p\delta_i^k\right] + \Lambda\delta_i^k \tag{6}$$

where

$$E_i^k = -F_{ia}F^{ka} + 1/4\delta_i^k F_{ab}F^{ab} \tag{7}$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \tag{8}$$

$$F^{ij};\, j = J^i \tag{9}$$

$$v^i v_i = 0, \qquad V_i V^i = 1 \tag{10}$$

Here p, ρ, σ, and Λ are, respectively, the pressure, density, radiation density, and the cosmological constant.

The appropriate forms of v^i and V^i are (Patel and Akabari, 1979)

$$v^i = \delta_1^i, \qquad V^i = (1/2L)^{1/2}\delta_4^i \tag{11}$$

We also assume that $2L$ is positive. We name the coordinates as

$$x^1 = r, \qquad x^2 = \alpha, \qquad x^3 = \beta, \qquad x^4 = u$$

We note that the nonzero components of R_i^k for the metric (5) are

$$R_1^1 = -\left[L_{rr} + \frac{2L_r}{b}\cot\left(\frac{r}{b}\right) - \frac{4L}{b^2}\right]$$
$$R_4^4 = -\left[L_{rr} + \frac{2L_r\cot(r/b)}{b}\right] \tag{12}$$
$$R_2^2 = R_3^3 = \left[\frac{(1-2L)\operatorname{cosec}^2(r/b)}{b^2} + \frac{4L}{b^2} - \frac{2L_r\cot(r/b)}{b}\right]$$
$$R_4^1 = \frac{-2L_u\cot(r/b)}{b}$$

Here and in what follows a suffix denotes partial derivatives (e.g., $L_r = \partial L/\partial r$, etc.)

Following the arguments similar to those made by Tupper we have to consider the following two cases only.

Case I:

$$F_{12} = F_{13} = F_{24} = F_{34} = 0$$

and at least one of F_{14}, F_{23} nonzero

Case II:

$$F_{12} = F_{13} = F_{14} = F_{23} = 0$$

and at least one of F_{24}, F_{34} nonzero

For the sake of brevity, we are not repeating here the arguments made by Tupper.

2. CASE I

This case is essentially the same as that discussed by Patel and Akabari. If both F_{14} and F_{23} are nonzero, then their solution is modified by the addition of a term representing magnetic monopole.

In this case the Maxwell equations (8) and (9) give

$$F_{14} = \frac{e(u)}{b^2}\operatorname{cosec}^2\frac{r}{b}, \qquad F_{23} = k\sin\alpha \tag{13}$$

and

$$J^i = \left(-\frac{e_u}{b^2}\operatorname{cosec}^2\frac{r}{b}, 0, 0, 0\right) \tag{14}$$

where $e(u)$ is an arbitrary function of u and k is a constant. It can be easily seen that J^i is a null vector. Using (13), (7), (10), (11), and (12) in (6) we obtain

$$L_{rr} + \frac{1-2L}{b^2}\operatorname{cosec}^2\frac{r}{b} - \frac{8\pi}{b^4}(e^2+k^2)\operatorname{cosec}^4\frac{r}{b} = 0 \tag{15}$$

$$8\pi p = -\Lambda + \frac{4\pi}{b^4}\operatorname{cosec}^4\frac{r}{b}(e^2+k^2) - \left(L_{rr} + \frac{2L_r}{b}\cot\frac{r}{b} + \frac{2L}{b^2}\right) \tag{16}$$

$$8\pi(p+\rho) = 4L/b^2 \tag{17}$$

and

$$8\pi\sigma = \frac{2L_u}{b}\cot\frac{r}{b} \tag{18}$$

It is painless to see that the solution of the differential equation (15) is

$$2L = 1 - \frac{2m}{b}\cot\frac{r}{b} + \frac{4\pi}{b^2}(e^2+k^2)\left(\cot^2\frac{r}{b} - 1\right) \tag{19}$$

where m is an arbitrary function of u. With this expression of $2L$ the expressions for p and σ become

$$8\pi p = -\Lambda - 2L/b^2 \tag{20}$$

$$8\pi\sigma = -\frac{2m_u}{b}\cot\frac{r}{b} + \frac{8\pi e e_u}{b^2}\left(\cot^2\frac{r}{b} - 1\right) \tag{21}$$

The final form of the metric is

$$(ds)^2 = 2du\,dr - b^2\sin^2\left(\frac{r}{b}\right)(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$
$$+\left[1 - \frac{2m}{b}\cot\frac{r}{b} + \frac{4\pi}{b^2}(e^2+k^2)\left(\cot^2\frac{r}{b} - 1\right)\right](du)^2 \tag{22}$$

when $k=0$, the metric (22) reduces to that discussed by Patel and Akabari. When $k=0$ and b tends to infinity the metric (22) reduces to that discussed by Bonner and Vaidya (1970). Also when e and m are constants and $k=0$ the metric (22) reduces to the Nordstrom metric in the cosmological background of Einstein's universe.

In the absence of the source (i.e., when $m = e = k = 0$, the metric (22) reduces to the metric (2) of Einstein's universe.

3. CASE II

In this case the Maxwell's equations (8) and (9) give

$$\frac{\partial F_{24}}{\partial r} = \frac{\partial F_{34}}{\partial r} = 0 \tag{23}$$

$$\frac{\partial F_{24}}{\partial \beta} = \frac{\partial F_{34}}{\partial \alpha} \tag{24}$$

and

$$J^{1} b^2 \sin^2 \frac{r}{b} = F_{24} \cot \alpha + \frac{\partial F_{24}}{\partial \alpha} + \frac{\partial F_{34}}{\partial \beta} \operatorname{cosec}^2 \alpha \tag{25}$$

Here also J^i is a null vector. The differential equation for the function $2L$ is

$$L_{,rr} + \frac{1 - 2L}{b^2} \operatorname{cosec}^2 \frac{r}{b} = 0 \tag{26}$$

The solution of (26) can be easily seen to be

$$2L = 1 - \frac{2m}{b} \cot \frac{r}{b} \tag{27}$$

where m is an arbitrary function of u. In this case the expressions for p, ρ, and σ become

$$8\pi p = -\Lambda - 2L/b^2 \tag{28}$$

$$8\pi \rho = \Lambda + 6L/b^2 \tag{29}$$

$$\sigma = \left[-\frac{m_u}{4\pi} \cos^2 \frac{r}{b} - F_{24} F_{24} - F_{34} F_{34} \operatorname{cosec}^2 \alpha \right] \frac{\operatorname{cosec}^2 (r/b)}{b^2} \tag{30}$$

Suppose that m_u is negative (i.e., the Schwarzschild mass is decreasing). Putting $\alpha^2(u) = -m_u/4\pi$, Tupper has given some solutions of (23) and

(24). Solution (a):

$$F_{24} \equiv \alpha(u) \cos\beta \cos\alpha$$
$$F_{34} \equiv -\alpha(u) \sin\beta \sin\alpha \tag{31}$$

For this solution J^1 and σ become

$$J^1 = -\frac{2\alpha(u)}{b^2} \operatorname{cosec}^2 \frac{r}{b} \sin\alpha \cos\beta$$

$$\sigma = \frac{\alpha^2(u)}{b^2}\left(-1 + \sin^2\alpha \cos^2\beta \operatorname{cosec}^2 \frac{r}{b}\right)$$

Solution (b):

$$F_{24} = \beta(u)$$
$$F_{34} = 0 \tag{32}$$

For this solution J^1 and σ are given by

$$J^1 = \frac{\beta(u)}{b^2} \cot\alpha \operatorname{cosec}^2 \frac{r}{b}$$

$$\sigma = \frac{\alpha^2(u) - \beta^2(u)}{b^2} \cot^2 \frac{r}{b} - \frac{\beta^2(u)}{b^2}$$

Solution (c):

$$F_{24} = \alpha(u) \sin\alpha$$
$$F_{34} = 0 \tag{33}$$

Here J^1 and σ become

$$J^1 = \frac{2\alpha(u)}{b^2} \cos\alpha \operatorname{cosec}^2 \frac{r}{b}$$

$$\sigma = \frac{\alpha^2(u)}{b^2}\left(\cos^2\alpha \cot^2 \frac{r}{b} - \sin^2\alpha\right)$$

When b tends to infinity in the solutions (a), (b), and (c) we recover the results obtained by Tupper. The metric for case II is

$$(ds)^2 = 2du\,dr - b^2\sin^2\frac{r}{b}(d\alpha^2 + \sin^2\alpha\,d\beta^2) + \left(1 - \frac{2m}{b}\cot\frac{r}{b}\right)(du)^2 \quad (34)$$

When b tends to infinity (34) reduces to the metric

$$(ds)^2 = 2du\,dr - r^2(d\alpha^2 + \sin^2\alpha\,d\beta^2) + \left(1 - \frac{2m}{b}\right)(du)^2$$

It should be noted that the above metric is also discussed by Vaidya (1953) as a solution of Einstein–Maxwell equations (without null field).

In the absence of the source (i.e., $m = 0$) the metric (34) reduces to the metric (2) of Einstein's universe.

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